ON DIFFERENTIAL EQUATIONS WITH PERIODIC INTEGRALS.

By DR. G. W. HILL, Washington, D. C.

The independent variable being conceived as time, a system of differential equations may be said to admit periodic integrals when the values of the dependent variables either exactly, or with approximate tendency, after a certain lapse of time, repeat their series of values. In the latter case the larger the lapse is made the more nearly is the repetition brought about. Strange as it may seem, this subject, except in the case of simply periodic integrals, is, at present, not completely understood. The text-books on differential equations are almost wholly engaged with the cases in which, by certain artifices, the integration can be accomplished in finite terms or reduced to quadratures. In the treatment of physical problems, however, equations of this sort are rarely met with. Far more frequently it is found that methods of approximation must be resorted to. Cauchy appears to be the author who has done most for the elucidation of this part of the subject. His memoirs are in his later *Exercises* and in the volumes of the *Comptes Rendus* for 1856 and 1857. In this article I propose to show how simply periodic integrals arise and afterwards to illustrate the general theory by treating a problem relating to the motion of a system of points.

I.

Having the independent variable t, and the two dependent variables x and x_1 let us suppose the latter satisfy the equations

$$\frac{dx}{dt} = x_1, \quad \frac{dx_1}{dt} = \mathrm{f}(x).$$

A cross multiplication between the members of these equations gives

$$x_1 \frac{dx_1}{dt} = \mathrm{f}(x) \frac{dx}{dt}.$$

The integral of this is, C being the arbitrary constant,

$$x_1^2 = 2\int \mathrm{f}(x)\, dx + C.$$

The values of x and x_1 being known for a given value of t, we readily find the value of C proper to the special case we treat. By substituting the value of x derived from this equation in the first of the differential equations we get

$$\frac{dx}{dt} = \sqrt{\left[2\int \mathrm{f}(x)\, dx + C\right]}.$$

The expression under the radical sign is a function of x; calling it X, let us consider the equation $X = 0$. Since real values of x are supposed to correspond to all values of t, X can never be negative; and, from the way the constant C was determined, it is plain that, for the given value of t, X is positive. Then in $X = 0$, let x be supposed to increase until a value $x = b$ is reached for which $X = 0$, that is to say a real root of this equation. Similarly let x diminish from the same point until a value $x = c$ is reached for which again $X = 0$, that is a second real root Then, X being positive for all values of x which lie between c and b, if the latter are non-multiple roots, X is negative for values of x which lie just outside these limits. Thus x must necessarily remain within the limits c and b. Also, in its motion, it always attains them; for suppose x is augmenting, then the radical, which forms the value of dx / dt, must be taken positively, and, from the law of continuity, must continue to be so taken until it becomes zero, that is until x arrives at the value b. But dx / dt cannot be positive beyond this point, for x cannot surpass b. Hence, after this, the radical must receive the negative sign, and, consequently, x begins to diminish. Again, from the law of continuity, this diminution is kept up until x has arrived at the value c. At this point the diminution must change into an augmentation, for x cannot fall below c. Thus the movement of x is a continuous swinging back and forth between the limits c and b.

We can put $$X = \frac{(b - x)(x - c)}{R^2},$$

R being a function of x which remains constantly positive and finite for all values of x between c and b. We can then write

$$\frac{dt}{dx} = \frac{R}{\sqrt{[(b - x)(x - c)]}}.$$

A new variable u can now be advantageously introduced in place of x. Let

$$x = a(1 - e \cos u),$$

where $a = \frac{1}{2}(b + c)$, and $e = (b - c) / (b + c)$; and u is equivalent to an integral number of circumferences when $x = c$, and augments by half a circumference when x, next following, attains the value b. Thus u, like t, augments continuously. We have

$$b - x = ae(1 + \cos u), \quad x - c = ae(1 - \cos u),$$

$$\sqrt{[(b - x)(x - c)]} = ae \sin u,$$

$$dx = ae \sin u \, du.$$

Therefore $$dt = R du.$$

As R is a one-valued function of x or of $a\,(1 - e \cos u)$, it can be expanded in the following periodic series

$$R = \frac{1}{n}\left[1 + \alpha_1 \cos u + 2\alpha_2 \cos 2u + 3\alpha_3 \cos 3u + \ldots\right],$$

n, α_1, α_2, etc. being constants, the first having the value

$$\frac{1}{n} = \frac{1}{\pi}\int_0^\pi R du.$$

Then c being an arbitrary constant,

$$n\,(t + c) = u + \alpha_1 \sin u + \alpha_2 \sin 2u + \alpha_3 \sin 3u + \ldots.$$

This series serves for determining t when x or u is given; but, more frequently it is x or u which is required in terms of t. It is necessary, then, to invert the series. The coefficients of the inverted series are most readily found by means of definite integrals. Let us suppose that it is required to find the periodic series, in terms of t, for a function of x and x_1 which we will denote by U. This function we assume to be always finite and continuous. The base of hyperbolic logarithms being ε, let us put

$$\zeta = n\,(t + c), \quad z = \varepsilon^{\zeta\sqrt{-1}}, \quad s = \varepsilon^{u\sqrt{-1}},$$

and, for brevity,

$$2S = \alpha_1 (s - s^{-1}) + \alpha_2 (s^2 - s^{-2}) + \alpha_3 (s^3 - s^{-3}) + \ldots.$$

The equation connecting z and s is

$$z = s\varepsilon^S.$$

We can suppose that $$U = \sum_{i=-\infty}^{i=+\infty} A_i z^i.$$

Then $$A_i = \frac{1}{2\pi}\int_0^{2\pi} U z^{-i} d\zeta = \frac{1}{2\pi}\int_0^{2\pi} U s^{-i} \varepsilon^{-iS}\, nR du.$$

U being $= F(x, x_1)$, we have

$$U = F\left\{a\,(1 - e \cos u),\quad \frac{ae \sin u}{R}\right\}$$

$$= F\left\{a\left[1 - \tfrac{1}{2}e\,(s + s^{-1})\right],\quad \frac{ae(s - s^{-1})}{2R\sqrt{(-1)}}\right\}.$$

Supposing that U is reduced to x, it is plain that the coefficient of z^i, in the

development of x in powers of z, is the same as the coefficient of s^i in the development of

$$a\left[1 - \tfrac{1}{2}e(s + s^{-1})\right]\left[1 + s\frac{\partial S}{\partial s}\right]\varepsilon^{-iS}$$

in powers of s.

By adopting the Besselian functions $J_\lambda^{(i)}$, we have

$$\varepsilon^{-\frac{1}{2}i\alpha_1(s - s^{-1})} = \sum_{i=-\infty}^{i=+\infty} J^{(j)}_{-\frac{1}{2}i\alpha_1} s^j, \quad \varepsilon^{-\frac{1}{2}i\alpha_2(s^2 - s^{-2})} = \sum_{j=-\infty}^{j=+\infty} J^{(j)}_{-\frac{1}{2}i\alpha_2} s^{2j}, \text{ etc.};$$

and the expression, given above, can be written

$$a\left\{\begin{array}{l} 1 - \frac{1}{2}\alpha_1 e \\ \quad + \left(\frac{\alpha_1 - e}{2} - \frac{2}{2.2}\alpha_1 e\right)(s + s^{-1}) \\ \quad + \left(\frac{2}{2}\alpha_2 - \frac{\alpha_1 e}{2.2} - \frac{3}{2.2}\alpha_3 e\right)(s^2 + s^{-2}) \\ \quad + \left(\frac{3}{2}\alpha_3 - \frac{2}{2.2}\alpha_2 e - \frac{4}{2.2}\alpha_4 e\right)(s^3 + s^{-3}) \\ \quad + \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

$$\times\,(\Sigma_j . J_{\lambda_1}^{(j)} s^j) . (\Sigma_j . J_{\lambda_2}^{(j)} s^{2j}) . (\Sigma_j . J_{\lambda_3}^{(j)} s^{3j}) \ldots,$$

where $\lambda_j = -\frac{1}{2}i\alpha_j$.

However, unless the coefficients $\alpha_1, \alpha_2, \alpha_3, \ldots$ decrease rapidly, this will not be a practical method of developing x in a periodic series. Generally it will be shorter to employ mechanical quadratures in obtaining the value of the definite integral. Let us suppose that

$$x = \tfrac{1}{2}\beta_0 + \beta_1 \cos\zeta + \beta_2 \cos 2\zeta + \beta_3 \cos 3\zeta + \ldots.$$

Then

$$\beta_i = \frac{2}{\pi}\int_0^\pi x \cos i\zeta\, d\zeta$$

$$= \frac{2}{\pi}\int_0^\pi a(1 - e\cos u)\left[1 + \alpha_1\cos u + 2\alpha_2\cos 2u + 3\alpha_3 \cos 3u + \ldots\right]\cos i\zeta\, du,$$

where, to obtain the value of ζ corresponding to a given value of u, we employ the equation

$$\zeta = u + \alpha_1 \sin u + \alpha_2 \sin 2u + \ldots.$$

It will be seen that this method is applicable to a much wider range of questions than the motion of planets in elliptic orbits. And the superiority of the

method of definite integrals over Lagrange's Theorem for the inversion of the series is quite manifest.

II.

In order to illustrate the preceding general theory, let us treat the problem of n material points moving about a centre under the action of central forces admitting a potential which is a function of the sum of the squares of the *radii vectores*. Each point will then move in a fixed plane and its *radius vector* will describe equal areas in equal times. Thus all will be virtually known in reference to these motions, provided we are able to express the *radii vectores* as functions of the time.

Let the *radii* be denoted as $r_1, r_2, \ldots r_n$, and the orbit longitudes, measured each from any point in its plane, as $\lambda_1, \lambda_2, \ldots \lambda_n$. For brevity, put

$$\rho^2 = r_1^2 + r_2^2 + \ldots + r_n^2.$$

Then, if the potential is represented by $\mathrm{f}(\rho)$, we shall have the two equations, representing generally all the equations of the problem,

$$\frac{d^2 r_i}{dt^2} - r_i \frac{d\lambda_i^2}{dt^2} = \mathrm{f}'(\rho)\frac{r_i}{\rho},$$

$$\frac{d\lambda_i}{dt} = \frac{h_i}{r_i^2},$$

h_i being the constant of areolar velocity. Consequently if we put

$$\Omega = \mathrm{f}(\rho) - \tfrac{1}{2}\Sigma\frac{h_i^2}{r_i^2},$$

the general form of the differential equations determining the *radii vectores* will be

$$\frac{d^2 r_i}{dt^2} = \frac{\partial \Omega}{\partial r_i}.$$

They have the integral, corresponding to that of living forces,

$$\Sigma\frac{dr_i^2}{dt^2} = 2(\Omega + C),$$

C being an arbitrary constant. Also we may derive

$$\Sigma r_i \frac{d^2 r_i}{dt^2} = \Sigma\, r_i \frac{\partial \Omega}{\partial r_i}.$$

By adding the last two equations,

$$\frac{d}{dt}\left(\rho\frac{d\rho}{dt}\right) = 2\mathrm{f}(\rho) + \rho\mathrm{f}'(\rho) + 2C,$$

an equation involving only the dependent variable ρ. Multiplying it by the factor $2\rho \frac{d\rho}{dt}$, and integrating, we get, A being an arbitrary constant,

$$\rho^2 \frac{d\rho^2}{dt^2} = 2\rho^2 [\mathrm{f}(\rho) + C] - A^2.$$

Whence
$$t + c = \int \frac{\rho d\rho}{\sqrt{\{2\rho^2 [\mathrm{f}(\rho) + C] - A^2\}}}.$$

Inverting this we shall have ρ as a function of t.

By dividing the penultimate equation by ρ^2 and differentiating, we get

$$\frac{d^2\rho}{dt^2} = \frac{\mathrm{f}'(\rho)}{\rho}\rho + \frac{A^2}{\rho^3}.$$

The general equation determining r_i is

$$\frac{d^2 r_i}{dt^2} = \frac{\mathrm{f}'(\rho)}{\rho} r_i + \frac{h_i^2}{r_i^3}.$$

As ρ is now a known function of t, r_i is the only unknown in it, and, consequently, the equation by itself suffices for determining it. To put the equation in a form suitable for integration, let us eliminate $\mathrm{f}'(\rho)$ between the last two equations. We get

$$\frac{d\,(\rho dr_i - r_i d\rho)}{dt^2} = \frac{h_i^2}{r_i^3}\rho - \frac{A^2}{\rho^3} r_i,$$

or
$$\rho^2 \frac{d}{dt}\left[\rho^2 \frac{d}{dt}\left(\frac{r_i}{\rho}\right)\right] = \left[\frac{h_i^2 \rho^4}{r_i^4} - A^2\right]\frac{r_i}{\rho}.$$

To simplify this, we will adopt an auxiliary variable ψ, such that

$$d\psi = \frac{A}{\rho^2}dt = \frac{A d\rho}{\rho\sqrt{\{2\rho^2 [\mathrm{f}(\rho) + C] - A^2\}}}.$$

Then
$$\frac{d^2\left(\frac{r_i}{\rho}\right)}{d\psi^2} = \left[\frac{h_i^2}{A^2}\frac{\rho^4}{r_i^4} - 1\right]\frac{r_i}{\rho}.$$

Whence, by integration, we derive

$$\left\{\frac{d\left(\frac{r_i}{\rho}\right)}{d\psi}\right\}^2 = 2\mathrm{a}_i - \frac{r_i^2}{\rho^2} - \frac{h_i^2}{A^2}\frac{\rho^2}{r_i^2},$$

a_i being the arbitrary constant. By putting

$$\frac{r_i}{\rho} = \sqrt{(u_i)},$$

we get
$$d\psi = \frac{du_i}{2\sqrt{\left[2\text{a}_i u_i - u_i^2 - \dfrac{h_i^2}{A^2}\right]}}.$$

For convenience, adopting a new constant e_i, in place of h_i, such that $h_i^2 / A^2 = \text{a}_i^2 (1 - e_i^2)$ the quantity under the radical sign becomes

$$[\text{a}_i (1 + e_i) - u_i] [u_i - \text{a}_i (1 - e_i)].$$

Thus, putting $u_i = \text{a}_i (1 - e_i \cos \varepsilon_i)$, ε_i being a new variable, we get $d\psi = \frac{1}{2} d\varepsilon_i$, and thus $\varepsilon_i = 2\psi + \alpha_i$, α_i being a constant. Thus we have, in fine,

$$\frac{r_i}{\rho} = \sqrt{\{\text{a}_i [1 - e_i \cos (2\psi + 2\alpha_i)]\}}.$$

As we have
$$\Sigma \frac{r_i^2}{\rho^2} = 1,$$

the constants a_i, e_i, and α_i satisfy the relations

$$\Sigma\, \text{a}_i = 1, \quad \Sigma\, \text{a}_i e_i \cos 2\alpha_i = 0, \quad \Sigma\, \text{a}_i e_i \sin 2\alpha_i = 0.$$

We thus have $2n$ independent arbitrary constants introduced by integration; the number there should be.

In order to find an expression for the longitudes, we take the general equation

$$d\lambda_i = \frac{h_i dt}{\text{a}_i \rho^2 [1 - e_i \cos 2 (\psi + \alpha_i)]}$$
$$= \frac{\sqrt{(1 - e_i^2)}\, d\psi}{1 - e_i \cos 2 (\psi + \alpha_i)}.$$

The integral of which gives

$$\tan (\lambda_i + \beta_i) = \sqrt{\left(\frac{1 + e_i}{1 - e_i}\right)} . \tan (\psi + \alpha_i),$$

β_i being the arbitrary constant.

To simplify the equations which give $t + c$ and ψ, we suppose that $\text{a}(1 + e)$ is the maximum value of ρ, and $\text{a}(1 - e)$ its minimum value. Then we can adopt a variable ε such that

$$\rho = \text{a} (1 - e \cos \varepsilon).$$

Thus $d\rho = \text{a}e \sin \varepsilon\, d\varepsilon$, and we may put

$$2\rho^2 [\text{f} (\rho) + C] - A^2 = R^2 \text{a}^2 e^2 \sin^2 \varepsilon,$$

where R remains constantly positive throughout the motion of ρ. Then

$$t + c = \int \frac{\rho}{R} d\varepsilon,$$

$$\psi = \int \frac{A}{\rho R} d\varepsilon.$$

R, being a function of ρ, is also one of $a(1 - e\cos\varepsilon)$, and thus is capable of being expanded in a converging series of terms, each consisting of a constant multiplied by the cosine of a multiple of ε. Also ρ / R and $A / \rho R$ can be expanded in similar series. Then the period T, in which ρ goes through the round of its values, is given by the definite integral

$$T = \int_0^{2\pi} \frac{a(1 - e\cos\varepsilon)}{R}\, d\varepsilon,$$

and the augmentation of the variable ψ, in the same time, will be equivalent to the definite integral

$$\int_0^{2\pi} \frac{A\,d\varepsilon}{a(1 - e\cos\varepsilon)R}.$$

If the value of the latter is 2π, ψ will augment by a circumference while ρ goes through its period. This is the case when $f(\rho) = \mu / \rho$; but, in general, this condition is not fulfilled.

Provided that A^2 is a positive quantity, it is plain that, after ψ has gone through its period, the longitudes and latitudes, whether as seen from the centre or from any of the points, all return to the same values. The same thing is true of the ratios of the *radii vectores*. Thus the movement of the system may be conceived as taking place under the operation of two distinct causes. The first producing a revolution of all the points about the centre in closed curves and in the same time, while the second, having a different period, changes the scale of representation of the system in space.

In the preceding treatment we have supposed that A^2 is a positive quantity. When this is not the case, some modifications must be made. Let us suppose first that $A = 0$. Then we have

$$t + c = \int \frac{d\rho}{\sqrt{\{2[f(\rho) + C]\}}},$$

and we may assume

$$\psi = \int \frac{d\rho}{\rho^2\sqrt{\{2[f(\rho) + C]\}}}.$$

Then

$$\left\{ \frac{d\left(\frac{r_i}{\rho}\right)}{d\psi} \right\}^2 = a_i - h_i^2 \frac{\rho^2}{r_i^2},$$

$$\psi + \alpha_i = \sqrt{\left(a_i \frac{r_i^2}{\rho^2} - h_i^2\right)},$$

$$\frac{r_i}{\rho} = \sqrt{\left\{\frac{(\psi + \alpha_i)^2 + h_i^2}{a_i}\right\}}.$$

Also
$$d\lambda_i = \frac{h_i dt}{r_i^2} = h_i \frac{\rho^2}{r_i^2} d\psi$$
$$= \frac{h_i \mathrm{a}_i d\psi}{(\psi + \alpha_i)^2 + h_i^2},$$
$$\tan(\lambda_i + \beta_i) = \frac{\mathrm{a}_i}{h_i}(\psi + \alpha_i).$$

In the second place, let A^2 be negative. Here it is only necessary in some places to accomplish the integrations by the aid of hyperbolic cosines instead of circular.

The differential equations of this problem, in the case where the *radii* are supposed to describe no areas, were first integrated by Binet.* But the addition, to the forces, of the terms arising from centrifugal action, much enhances the interest of the problem.

*See Liouville, *Journal de Mathématiques*, First Ser. Tome II. p. 457.

ON THE FOCAL CHORD OF A PARABOLA.

By PROF. R. H. GRAVES, Chapel Hill, N. C.

Let $y^2 = 4ax$ be the equation to a parabola, s its focus, and PSP' a focal chord. Let the tangent and normal at P' meet the diameter through P at M and N.

It may be easily proved that $PM = PN = PP'$ and that a similar property holds for the tangent and normal at P.

Therefore, if two equal rhombs be constructed on PP' having two other sides of each parallel to the axis, their diagonals are tangents and normals at P and P'; and the tangent at one point is parallel to the normal at the other.

Each normal chord divides the other in the ratio 1 : 3.

The chord joining the other ends of the normal chords is parallel to PP' and three times as long.

A line perpendicular to PP' at S, and terminated by this parallel chord and the pole of PP', is divided by S in the ratio 1 : 4.

Hence the locus of the foot of the perpendicular dropped from S on the parallel chord is a right line, whose equation is $x = 9a$.

Hence the envelope of the parallel chord is a confocal parabola, having for its equation $y^2 = 32a(9a - x)$.

It cuts the original parabola orthogonally where it is cut by its evolute.